

Fuzzy Orbifolds

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Abstract

A family of fuzzy orbifolds are generated by looking at sub-algebras of the fuzzy sphere. One of them is actually commutative and can be mapped exactly onto a lattice. The others are fuzzy approximations of S^2/Z_N where Z_N is the cyclic group of rotations of angle $2\pi/N$ and provides the first example of the “fuzzification” of a space with singularities (at the poles). This construction can easily be generalised to other fuzzy spaces.

1 Introduction

The idea behind a fuzzy space is to approximate the algebra of functions of a (commutative) space by a sequence of finite dimensional algebra, i.e. an algebra of matrices, of increasing dimension.

“Fuzzifying” a space can be done in at least two ways. The first one is to quantise the algebra of functions of a commutative space as a phase space. The archetypal example of this method is the fuzzy sphere [1], but complex projective spaces \mathbb{CP}^n and other coadjoint orbits can also be treated this way. In this case, the algebra of matrices can be embedded into the algebra of functions of the commutative space with a deformed product called the star product [2], the symmetries of the commutative space are exactly preserved on the matrix approximations, and derivations can be identified with infinitesimal transformations of the symmetry group. From there, any algebraic expression in the algebra of function can easily be approximated on the matrix algebras.

We are interested in field theories, and this kind of fuzzy space can then be used for non-perturbative studies [5]. The space can be described by a triple containing a sequence of algebras which reduces essentially to a sequence of dimensions of matrix spaces, a differential operator necessary to describe the kinetic part of the action (Laplacian for a scalar field, Dirac operator for a spinor field...) and a scalar product to produce a scalar action. With these ingredients, it is possible to describe field theories on the fuzzy space.

This is a very elegant method, but unfortunately it can only be applied to a limited number of spaces. Note however, that Cartesian products and “fuzzification” commute allowing one to “fuzzify” any Cartesian product of such spaces easily.

The second “fuzzification” method only requires that a functional integral on the commutative space be approximated by an integral on the matrices. This is a much weaker requirement which is sufficient for the purpose of studying field theory, and allows one to “fuzzify” a much larger class of spaces [3]. The trick is generally to select a linear sub-space with the right finite dimensional representation of the symmetry group, and then perform a kind of compactification on the additional degrees of freedom by putting an addition weight in the functional integral. Again, the necessary derivation operators can be associated with infinitesimal transformations of the symmetry group. In this case, the triple can also be defined, but does not describe all the structure since the functional integration measure must also be added to it.

Although this latter method allows for a much larger class of spaces, even including spaces with edges [4], it would appear to be limited to smooth, non-singular, spaces.

In this paper, we observe that there are sub-algebras of the fuzzy sphere stable under the Laplacian operator, which can therefore be used to “fuzzify” scalar field actions on subspaces of the algebra of

functions. In this case, this sub-algebra can be identified with the algebra of functions of the orbifold S^2/Z_n with Z_n the cyclic group with N elements. Similar results can certainly be generalised to more complicated fuzzy spaces such as the complex projective planes \mathbb{CP}^n without difficulty.

In the next Section, we introduce the fuzzy sphere before introducing in Section three its sub-algebras which give rise to the fuzzy orbifolds. These sub-algebras are actually split between a commutative sub-algebra which, together with its Laplacian, can be mapped onto a standard lattice, and non-commutative sub-algebras. Finally, we conclude in Section four.

2 The fuzzy sphere

The simplest example of a fuzzy space is the fuzzy sphere [1]. As explained in the Introduction, for the purpose of studying a scalar field theory, the only ingredient required to fix the geometry is a Laplacian operator and a scalar product on each matrix algebra. Since derivations on the commutative sphere can be viewed as infinitesimal $SU(2)$ transformations, the Laplacian on a $(2s+1) \times (2s+1)$ matrix algebra, denoted Mat_{2s+1} , is proposed as

$$\mathcal{L}^2 \phi = [L_i, [L_i, \phi]], \quad (1)$$

where L_i are the angular momentum operators in the $2s+1$ dimensional irreducible representation of $SU(2)$. The canonical matrix scalar product

$$\langle \phi | \psi \rangle = \frac{4\pi}{2s+1} \text{Tr}(\phi^\dagger \psi). \quad (2)$$

is chosen with a multiplicative coefficient such that the unit matrix has the same norm as the unit function on the sphere of radius one.

The spectrum and eigenmatrices of the proposed Laplacian can be recognised from the adjoint action of angular momentum as

$$\mathcal{L}^2 \hat{Y}_{lm} = l(l+1) \hat{Y}_{lm}, \quad 0 \leq l \leq 2s \quad (3)$$

$$[L_3, \hat{Y}_{lm}] = m \hat{Y}_{lm}, \quad 0 \leq |m| \leq l, \quad (4)$$

where the matrices \hat{Y}_{lm} are the polarisation tensors whose normalisation is defined according to the chosen scalar product

$$\frac{4\pi}{2s+1} \text{Tr}(\hat{Y}_{lm}^\dagger \hat{Y}_{lm}) = 1. \quad (5)$$

This is precisely the spectrum of the Laplacian on the commutative sphere truncated at angular momentum $2s$, thus vindicating this choice.

A clean way of recognising the approximation of a sphere in these matrix algebras is to introduce a mapping which associates a function on the sphere with each matrix of the algebra Mat_{2s+1} and pulls back most of the structure on the algebra of functions of the sphere onto the matrix algebra. There are various ways to define such a mapping, such as using coherent states [7] or the Brezin symbol map [6]. For instance, the latter is given by

$$\mathcal{M}_s : \text{Mat}_{2s+1} \rightarrow \mathcal{C}^\infty(S^2) \quad (6)$$

$$M = \sum_{l=0}^{2s} \sum_{m=-l}^l c_{lm} \hat{Y}_{lm} \mapsto f(\mathbf{n}) = \sum_{l=0}^{2s} \sum_{m=-l}^l c_{lm} Y_{lm}(\mathbf{n}), \quad (7)$$

where the functions $Y_{lm}(\mathbf{n})$ are the usual spherical harmonics on the sphere, i.e. the eigenvectors of the Laplacian operator on the sphere. By definition, this mapping \mathcal{M}_s is linear and maps the Laplacian \mathcal{L}^2 on Mat_{2s+1} onto the Laplacian on the sphere. In fact, the three derivatives on the sphere $\nabla_l = i\varepsilon_{jkl}x_j\partial_k$ are pulled back to simple derivations on the matrix algebra given by

$$\mathcal{L}_i \phi = [L_i, \phi]. \quad (8)$$

By construction, the action of the group $SU(2)$ is preserved on both sides. Furthermore, since the eigenvectors of the Laplacian on the matrix space and on the sphere form orthonormal bases on their respective spaces, this mapping is an injective isometry. Its image, on which the mapping is one to one, $\mathcal{M}_s(\text{Mat}_{2s+1})$ is given by all the functions with angular momentum only up to $2s$ and form a sequence of increasing (for the inclusion) sets which become dense in $\mathcal{C}^\infty(S^2)$ in the limit of infinite matrices. The matrix product is mapped to a (non-commutative) product of functions on the sphere called a $*$ -product

$$\mathcal{M}_s(\phi\psi)(\mathbf{n}) = (\mathcal{M}_s(\phi) *_s \mathcal{M}_s(\psi))(\mathbf{n}), \quad (9)$$

which is evidently distinct from the usual (commutative) product of functions. It is possible to verify that in the limit of infinite matrices $s \rightarrow \infty$, the star product tends to the usual product. More precisely, for $(f_s, g_s) \in (\mathcal{M}_s(\text{Mat}_{2s+1}))^2$ two functions with angular momentum truncated at $2s$, and $t \geq s$,

$$(f_s *_t g_s)(\mathbf{n}) = f_s(\mathbf{n})g_s(\mathbf{n}) + \mathcal{O}\left(\frac{1}{t}\right). \quad (10)$$

Note in passing that complex conjugation of a function on the sphere pulls back to hermitian conjugation on the matrix algebra. Consequently, as proposed in the introduction, real functions pull back to hermitian matrices. Similarly, integration on the sphere which is similar to scalar product with the unit function pulls back to the trace on the matrix algebra.

Thus, in the limit when s goes to infinity, the mapping \mathcal{M}_s becomes an isomorphism of algebras which preserves rotational invariance, the Laplacian and the scalar product (2). This proves that the fuzzy spaces, as defined by the triple $(\text{Mat}_{2s+1}, \mathcal{L}^2, \langle \cdot, \cdot \rangle)$ go over to the sphere in the limit of infinitely large matrices.

Another mapping with similar properties which is often introduced is the one obtained by looking at the diagonal elements of a matrix in a coherent states representation. Compared to \mathcal{M}_s , this mapping trades the isometry property for the conservation of the notion of state, in the sense that it maps a state of Mat_{2s+1} into a state of $\mathcal{C}^\infty(S^2)$. More generally it conserves the notion of positivity in the sense that a positive matrix is sent to a positive function. In this case, the corresponding star product can also be expressed in a simple exact form [7].

This introduction to the fuzzy sphere described spheres of radius one. Getting spheres of different radius R , is just a matter of scaling the scalar product (2) and Laplacian (1) appropriately:

$$\mathcal{L}^2 \rightarrow \frac{1}{R^2} \mathcal{L}^2, \quad (11)$$

$$\frac{4\pi}{2s+1} \text{Tr}(\phi^\dagger \psi) \rightarrow \frac{4\pi R^2}{2s+1} \text{Tr}(\phi^\dagger \psi). \quad (12)$$

Since this is such a simple generalisation, only spheres of radius one will be considered in the following.

With the fuzzy sphere cleanly defined, its sub-algebras can now be investigated.

3 sub-algebras of the fuzzy sphere

As mentioned and illustrated in the introduction in the case of the sphere, a triple of a sequence of algebras of diverging dimension, a Laplacian and a scalar product is sufficient to define a fuzzy space on which a scalar field theory can be studied.

It is clear that if a sequence of matrix sub-algebras stable under the action of the Laplacian can be found, a similar triple will immediately be induced on them. Stability under hermitian conjugation is not necessary but will also be retained in our examples. In a way, the construction here can be viewed as a method of defining a “fuzzy sub-space”.

In the case of the fuzzy sphere, such a sub-algebra must be generated by a family of eigenvectors of the Laplacian \hat{Y}_{lm} closed under multiplication and hermitian conjugation, or equivalently under $m \rightarrow -m$.

3.1 The commutative sub-algebra

The simplest such example one can imagine is given by the sub-algebra of diagonal matrices which will be called \mathcal{A}_0^s . These are generated exactly by the polarisation operators \hat{Y}_{l0} . To see this, note first that these two spaces have the same dimension $2s + 1$, and that the \hat{Y}_{l0} are themselves diagonal since, using (4), for $i \neq j$,

$$(\hat{Y}_{l0})_{ij} = \frac{1}{i-j}(\mathcal{L}_3 \hat{Y}_{l0})_{ij} = 0. \quad (13)$$

Thus, in the limit where s tends to infinity, the mapping \mathcal{M}_s defined in Eq. (7) sends this algebra \mathcal{A}_0^s to the algebra of functions \mathcal{A}_0^∞ on the sphere invariant under rotations around the third axis

$$\mathcal{A}_0^\infty = \{\phi(\theta, \phi) | \phi(\theta, \phi) = \phi(\theta)\}, \quad (14)$$

and using the usual Laplacian on the sphere

$$-\Delta\phi = \frac{d}{\sin(\theta)d\theta}(\sin(\theta)\frac{d\phi}{d\theta}) \quad (15)$$

where (θ, ϕ) are the spherical coordinates. Equivalently, by changing variables from θ to $z = \cos(\theta)$, this space can be viewed as the space of \mathcal{C}^2 functions on the segment

$$V_0 = [-1, 1], \quad (16)$$

with the Laplacian

$$\Delta f = \frac{d}{dz}((1-z^2)\frac{df}{dz}), \quad (17)$$

and no constraints at the boundaries $z = \pm 1$.

In conclusion, the sequence of matrix sub-algebras \mathcal{A}_0^s is a “fuzzy subspace” of the fuzzy sphere which approximates a segment with the Laplacian (17).

Since the algebra \mathcal{A}_0^s is actually commutative, it should be possible to map it to an algebra of functions of a lattice. To see that, the degrees of freedom of the algebra \mathcal{A}_0^s must be identified with some lattice points, and the Laplacian with a finite difference Laplacian. The Laplacian on the discrete algebra

$$\mathcal{A}_0^s = \{\text{Diag}(\phi_i)_{-l \leq i \leq l} | \phi_i \in \mathbf{C}\}, \quad (18)$$

is given by the diagonal matrix

$$-(\mathcal{L}^2\phi)_{ii} = (l(l+1) - i^2)(\phi_{i+1} + \phi_{i-1} - 2\phi_i) - i(\phi_{i+1} - \phi_{i-1}). \quad (19)$$

It is clear that this Laplacian (19) is a finite difference approximation of the commutative Laplacian (17) on the lattice where $z = i/\sqrt{l(l+1)}$, that is

$$\phi_i = \phi(\arccos(i/\sqrt{l(l+1)})), \quad -l \leq i \leq l. \quad (20)$$

Note that the sub-algebras \mathcal{A}_0^s have no continuous symmetry left, and that the derivations defined on the fuzzy sphere by (8) do not close in \mathcal{A}_0^s except for \mathcal{L}_3 which is trivially zero. This is as it should be for a lattice.

Further sub-algebras of \mathcal{A}_0^s can easily be constructed by considering it as a lattice. However, they hold little interest from the point of view of fuzzy spaces.

3.2 Other orbifolds

More generally, the product of two polarisation operators can be expanded on the basis of polarisation operators itself in a complicated way which involves Clebsh–Gordan coefficients. However, it should be

noted that the Leibnitz rule implies that the axial quantum number is just added under multiplication, that is

$$\mathcal{L}_3(\hat{Y}_{lm}\hat{Y}_{kn}) = (m+n)\hat{Y}_{lm}\hat{Y}_{kn}. \quad (21)$$

Thus a generalisation of the sub-algebra \mathcal{A}_0^s is given by the sub-algebras \mathcal{A}_k^s , k a fixed positive integer, generated by the polarisation operators with axial quantum number a multiple of k

$$\mathcal{A}_k^s = \text{Span}(\hat{Y}_{lkm}, 0 \leq l \leq s, 0 \leq k|m| \leq l). \quad (22)$$

The notation \mathcal{A}_k^s adopted here is consistent with that of the space \mathcal{A}_0^s introduced in the previous subsection which does indeed correspond to the case when $k = 0$. More generally, if the requirement that the space be invariant under conjugation is dropped, spaces such as

$$\mathcal{A}_k^{\pm s} = \text{Span}(\hat{Y}_{lkm}, 0 \leq l \leq s, 0 \leq \pm km \leq l), \quad (23)$$

are also acceptable.

Using the same trick as in Eq. (13), it is possible to check that \hat{Y}_{lkm} has only non-zero entries on the off-diagonal $(i, i-m)$. Thus, the sub-algebra \mathcal{A}_k^s can be recognised as the space of band diagonal $(2s+1) \times (2s+1)$ matrices which are non-zero on the diagonal and then alternate $k-1$ diagonals of zeros with a non-zero diagonal, or written algebraically,

$$\mathcal{A}_k^s = \{M \in \text{Mat}_{2s+1} | M_{ij} = 0, i \not\equiv j \pmod{k}\}. \quad (24)$$

Again, these sequences of sub-algebras can be seen as fuzzy approximations of the algebra of functions on the sphere generated by the family of spherical harmonics Y_{lkm} . These in turn can be identified with the space of functions invariant under rotation around the third axis of angle $2\pi/k$, or conversely as the space of functions on the slice of the sphere defined by

$$V_k = \{(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) | 0 \leq \phi < 2\pi/k\}, \quad (25)$$

with periodic boundary conditions between $\phi = 0$ and $\phi = 2\pi/k$. This space of functions is equivalent to the orbifold S^2/Z_k , with Z_k the cyclic group with k elements.

Note that in this case as in the commutative case, the space \mathcal{A}_k^s is not closed under derivations \mathcal{L}_i as defined in (8) unless $k = 1$ which corresponds to the fuzzy sphere. On the other hand, this space retains a continuous $U(1)$ symmetry group corresponding to axial rotation around the third axis.

Of course, the algebra \mathcal{A}_k^s can be rewritten as a direct sum of matrix algebras itself

$$\mathcal{A}_k^s = \oplus_{i=0}^{k-1} E_i \otimes E_i, \quad E_i = \oplus_{0 \leq k|m|+i \leq l} |s, km+i\rangle, \quad (26)$$

where $|s, m\rangle$ with $0 \leq |m| \leq 2s+1$ is the canonical basis of the linear space on which $(2s+1) \times (2s+1)$ matrices act.

There does not appear to be other such examples on the fuzzy sphere. For instance, \mathcal{RP}^2 , whose algebra of function is generated by a family of spherical harmonics, the $Y_{2l,m}$, can not be fuzzified in this way because the corresponding family of polarisation operators, $\hat{Y}_{2l,m}$ is not stable under multiplication.

4 Conclusion

A family of “fuzzy sub-spaces” of the fuzzy sphere were proposed which are sub-algebras closed under the Laplacian so that a triple is automatically induced from the triple of their parent space. There is an obvious generalisation of this construction to other fuzzy spaces such as the complex projective planes. Although the fuzzy spheres of other dimensions and tori introduced in [3] were constructed in a different way than the fuzzy sphere, a similar construction can also be performed there.

The main difference with previously proposed fuzzy spaces is that the limiting commutative space is singular (at the poles) and that the algebra is a direct sum of matrix algebras instead of being a single matrix algebra. This suggests a possible relation between these two properties.

References

- [1] J. Madore, *Class. Quant. Grav.* **9**, 69 (1992).
- [2] A.P. Balachandran, B. Dolan, J. Lee, X. Martin and D. O'Connor *J.Geom.Phys.* **43** 184-204 (2002) and [hep-th/0107099](#); G. Alexanian, S. Pinzul and A. Stern *Nucl.Phys.* **B600/3**, 531 (2001) and [hep-th/0010187](#).
- [3] Julieta Medina and Denjoe O'Connor, "Scalar Field Theory on Fuzzy S^4 " *JHEP* **0311**, 051 (2003); Brian P. Dolan and Denjoe O'Connor, "A Fuzzy Three Sphere and Fuzzy Tori" *JHEP* **0310**, 060 (2003); Brian P. Dolan, Denjoe O'Connor and Peter Presnajder, "Fuzzy Complex Quadrics and Spheres" *JHEP* **0402**, 055 (2004).
- [4] A.P. Balachandran, K. Gupta and S. K rk o  lu, *JHEP* **0309** 007 (2003); F. Lizzi, P. Vitale and A. Zampini, *Jhep* **0308**, 057 (2003).
- [5] X. Martin, [hep-th/0402230](#), accepted by *Jhep*.
- [6] D. O'Connor, *Mod. Phys. Lett.* **A18**, Numbers 33–35, 2423-2430 (2003).
- [7] P. Pre najder, *J. of Math. Phys.* **41** 2789-2804 (2000); A.P. Balachandran, B.P. Dolan, J. Lee, X. Martin and D. O'Connor, *J. Geom. Phys.* **43** 184-204 (2002).